A Diffusion Approximation Analysis of an ATM Statistical Multiplexer with Multiple Types of Traffic Part II: Time-Dependent Solutions

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Abstract

We introduce a diffusion model to approximately characterize a statistical multiplexer for a heterogeneous set of traffic sources, which is a basis for ATM (asynchronous transfer mode) fast packet switching in the future B-ISDN (Broadband Integrated Services Digital Networks). Under a reasonable set of assumptions, this diffusion process can then be approximated by a multi-dimensional Ornstein-Uhlenbeck process, which is a Gaussian Markov process. The packet arrival process is shown to be a Gaussian (but not Markov) process, which adequately captures the correlated nature of packet arrivals and determine the statistical behavior of the buffer content. Some simulated sample paths and estimated correlation functions (or the correlograms) will be shown to verify the diffusion approximation. We then apply our analytical results to evaluate the multiplexer's dynamic behavior, i.e., the time-dependent packet loss probabilities and the transient periods at the cell and burst levels.

1 Introduction

In our earlier work [10], we developed a diffusion-process approximation model for an ATM statistical multiplexer with multiple types of information sources (i.e., multimedia-voice, data, image and video etc.). Essentially an aggregate "on-off" behavior of many sources of each traffic type can be approximately modeled in terms of a diffusion process —a continuous-time, continuous-path Markov process.

Furthermore, under a reasonable set of assumptions the diffusion equation becomes the one for the Ornstein-Uhlenbeck process [3, 7], a refined model of the Brownian motion. The O-U process is a Gauss-Markov process, therefore the aggregate packet traffic from different types of sources can be also approximated by a Gaussian (but not Markov) process. The behavior of the multiplexer buffer is closely related to the multi-dimensional O-U process, and the analysis of buffer behavior is reduced to the well studied differential equation for Weber's parabolic cylinder function. The joint probability distribution of the buffer content and "on-off" sources, in their equilibrium-state, is then given in terms of the corresponding eigenvalues and eigenfunctions, which are Hermite functions. For the derivation of our diffusion approximation model and its steady-state analysis, the reader is referred to [10] and references therein.

In present paper we extend the earlier result and derive time-dependent (i.e., transient) solutions of the underlying diffusion equation, then discuss its applications to performance analysis of an ATM network. In optical communication networks the transmission speed is so high that the ratio of propagation delay to packet transmission time becomes much greater than unity. This implies that the conventional steady-state analysis of network congestion may not apply in this case, since by the time information is provided to a network controller, the congestion situation may have changed significantly. Therefore, some kind of predictive or preventive control scheme must be developed, and better understanding of the transient behavior of network dynamics will be essential in developing such control algorithms.

2 Diffusion Approximation Model

Let there be N_k sources of type k, where $k = 1, 2, \cdots, K$, and let $J_k(t)$ denote the number of type k sources in "on" (or "burst") state: the remaining $N_k - J_k(t)$ sources are "off" (or "silent"). We assume that successive "on" and "off" periods of each source form an alternating renewal process. Let $Y_k(t)$ be the diffusion process approximation of $J_k(t)$, the number of type k sources that are "on" ($J_k(t) = 0, 1, 2, \cdots, N_k; 1 \leq k \leq K$). While a type k source is on, it generates R_k [packets/sec.]. The average "on" and "off" periods of a type k source are $\frac{1}{\beta_k}$ and $\frac{1}{\alpha_k}$, respectively. Let $Q(t)$ be the fluid approximation for the number of outstanding packets in the multiplexer output. Then $Q(t)$ increases or decreases, de-

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pending on whether the aggregate packet arrival rate

$$R(t) = \sum_{k=1}^{K} R_k Y_k(t) \quad (1)$$

exceeds the multiplexer output link capacity C [packets/sec.] or not. Thus $Q(t)$ depends on the multi-dimensional diffusion process $\mathbf{Y}(t) = \{Y_k(t); 1 \leq k \leq K\}$ only through $R(t)$, i.e.,

$$\frac{dQ(t)}{dt} = \begin{cases} R(t) - C, & \text{when } R(t) > C \text{ or } Q(t) > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Although neither $R(t)$ nor $Q(t)$ is a Markov process, the multivariate process $(\mathbf{Y}(t), Q(t))$ is Markovian, and its stochastic behavior is governed by the following transition probability density function

$$\begin{aligned} & f(\mathbf{y}; x, t|\mathbf{y}_0, x_0)d\mathbf{y}dx \\ = \ & Prob[Y_k(t) = y_k, 1 \leq k \leq K; \text{ and } Q(t) = x \\ & |\mathbf{Y}(0) = \mathbf{y}_0, Q(0) = x_0]. \end{aligned} \quad (3)$$

Then we can show (see [10]) that the above function should satisfy the following partial differential equation:

$$\frac{\partial f}{\partial t} + (\sum_{k=1}^{K} R_k y_k(t) - C)\frac{\partial f}{\partial x} = \mathbf{L}f, \quad (4)$$

where we write, for brevity of notation, f instead of $f(\mathbf{y}; x, t|\mathbf{y}_0, x_0)$ and $\mathbf{L}$ refers to the infinitesimal generator of the multi-dimensional diffusion process $\mathbf{Y}(t)$ defined by

$$\mathbf{L}f = \sum_{k=1}^{K} \left\{ -\frac{\partial}{\partial y_k}[m_k(y_k, t)f] + \frac{1}{2}\frac{\partial^2}{\partial y_k^2}[v_k(y_k, t)f] \right\}. \quad (5)$$

where $m_k(y_k, t) = N_k\alpha_k - (\alpha_k + \beta_k)y_k$, and $v_k(y_k, t)$ is associated with the second order statistics of k-type sources.

Let us first consider the limit $x \to \infty$, and focus on the multivariate diffusion process $\mathbf{Y}(t)$.

By defining

$$b_k = \alpha_k + \beta_k, \ D_k = \frac{v_k(y_k^*, t)}{2} \text{ and } y_k^* = \frac{N_k\alpha_k}{\alpha_k + \beta_k}, \quad (6)$$

the transition probability density function of $\tilde{f}(\mathbf{y}; t|\mathbf{y}_0) = \lim_{x\to\infty} f(\mathbf{y}; t|\mathbf{y}_0, x_0)$ satisfies the following forward diffusion equation:

$$\frac{\partial \tilde{f}}{\partial t} = \tilde{\mathbf{L}}\tilde{f}, \quad (7)$$

where $\tilde{\mathbf{L}}$ is the operator defined by

$$\tilde{\mathbf{L}}\tilde{f} = \sum_{k=1}^{K} \left\{ b_k\frac{\partial}{\partial y_k}[(y_k - y_k^*)\tilde{f}] + D_k\frac{\partial^2}{\partial y_k^2}\tilde{f} \right\}. \quad (8)$$

Equation (7) represents the multi-dimensional *Ornstein-Uhlenbeck* process, which is obtained by assuming the linear drift coefficient $m_k(y_k, t) = -b_k(y_k - y_k^*)$ and the constant diffusion coefficient $v_k(y_k^*, t) = 2D_k$ in Eq.(5). Thus, each $Y_k(t)$ is characterized as an *Ornstein-Uhlenbeck* process, which satisfies the following stochastic differential equation [2]:

$$dY_k(t) = -b_k(Y_k(t) - y_k^*)dt + \sqrt{2D_k}dW_k(t) \quad (9)$$

where $W_k(t)$ is the standard Brownian motion.

If we express the joint density function $\tilde{f}$ in a product form

$$\tilde{f}(\mathbf{y}; t|\mathbf{y}_0) = \prod_{k=1}^{K} \tilde{f}_k(y_k, t|y_{k_0}), \quad (10)$$

then the multi-dimensional diffusion equation (7) can be split into K independent diffusion process equations:

$$\frac{\partial \tilde{f}_k}{\partial t} = b_k\frac{\partial}{\partial y_k}[(y_k - y_k^*)\tilde{f}_k] + D_k\frac{\partial^2 \tilde{f}_k}{\partial y_k^2}, \ k = 1, \cdots, K, \quad (11)$$

The solution of Eq.(11) has a Gaussian distribution with mean $m_k(t)$ and variance $\sigma_k^2(t)$, respectively, i.e.,

$$\tilde{f}_k(y_k, t|y_{k_0}) = \frac{1}{\sqrt{2\pi\sigma_k^2(t)}}exp(-\frac{(y_k - m_k(t))^2}{2\sigma_k^2(t)}), \quad (12)$$

where

$$m_k(t) = y_k^*(1 - e^{-b_k t}) + y_{k_0}e^{-b_k t}, \quad (13)$$

$$\sigma_k^2(t) = \frac{D_k}{b_k}(1 - e^{-2b_k t}). \quad (14)$$

As $t \to \infty$, the mean tends to y_k^* and the variance to $\sigma_k^{*^2}$.

It is known that the O-U process defined by Eq.(9) is the only Gauss-Markov process with stationary transition probabilities. Therefore, the autocovariance function of the O-U process $Y_k(t)$ is given by

$$cov_k(\tau) = \frac{D_k}{b_k}e^{-b_k\tau}. \quad (15)$$

It is worthwhile to point out that the covariance function of $J_k(t)$ has exactly the same expression as Eq.(15) if we assume exponential distributions for both "on" and "off" periods, although the binomial distribution of $J_k(t)$ is approximated by the Gaussian distribution of $Y_k(t)$.

Simulated sample paths of $J_k(t)$ and $Y_k(t)$ are shown in Figures 1 and 2, in which the O-U process $Y_k(t)$ approximates the type-k aggregate traffic $J_k(t)$. As we see, sample paths of the diffusion process $Y_k(t)$ characterize the statistical behavior of the staircase function $J_k(t)$, when the N_k, number of multiplexed sources, is moderately large, and their behaviors are almost indistinguishable when N_k becomes as large as 50. Although not presented here, we verified that the distribution of these processes approach the Gaussian distributions by plotting them in fractile diagrams [6]. In Figure 3 the estimated correlation function (sometimes called the correlogram, when normalized) of the observed data $Y_k(t)$ of (9) is compared with the analytical expression (15).

Note that the aggregate process $R(t)$ is a weighted sum of the K independent Gaussian processes $\{Y_k(t); 1 \le k \le K\}$, thus it is also a Gaussian process, which has the distribution

$$\tilde{f}_R(r,t|r_0) = \frac{1}{\sqrt{2\pi\sigma_R^2(t)}} exp\left\{-\frac{(r-m_R(t))^2}{2\sigma_R^2(t)}\right\}, \quad (16)$$

where

$$m_R(t) = \sum_{k=1}^{K} R_k m_k(t), \quad (17)$$

$$\sigma_R^2(t) = \sum_{k=1}^{K} R_k^2 \sigma_k^2(t), \quad (18)$$

$$cov_R(\tau) = \sum_{k=1}^{K} R_k^2 cov_k(\tau). \quad (19)$$

Now let us return to the generalized diffusion process equation (4). We approximate $\mathbf{L}$ by its O-U process operator $\tilde{\mathbf{L}}$ and take the Laplace transform with respect to time t, i.e., $f^*(\mathrm{y}; x, s|\mathrm{y}_0, x_0) = \mathcal{L}_t\{f(\mathrm{y}; x, t|\mathrm{y}_0, x_0)\}$. Then Eq.(4) becomes

$$sf^* - \delta(\mathrm{y}-\mathrm{y}_0)\delta(x-x_0) + (\sum_{k=1}^{K} R_k y_k - C)\frac{\partial f^*}{\partial x}$$
$$= \tilde{\mathbf{L}} f^*. \quad (20)$$

We assume that buffer is initially empty ($x_0 = 0$) and consider the non-empty buffer ($Q(t) > 0$) case by using the separation of variables technique, i.e,

$$f^*(\mathrm{y}; x, s|\mathrm{y}_0, 0) = h(s,x) \cdot g^*(s, \mathrm{y})$$
$$= h(s,x) \prod_{k=1}^{K} g_k^*(s, y_k). \quad (21)$$

By substituting Eq.(21) into Eq.(20), we have

$$\frac{h'(s,x)}{h(s,x)} = \frac{\tilde{\mathbf{L}} g^*(s,\mathrm{y}) - sg^*(s,\mathrm{y})}{(\sum_{k=1}^{K} R_k y_k - C) g^*(s,\mathrm{y})} = u(s), \quad (22)$$

where $u(s)$ is a function of s and will be determined later.

From the first and the last terms in Eq.(22) we readily find

$$h(s,x) = e^{u(s)\cdot x}. \quad (23)$$

For arbitrary s, we partition the link capacity C as

$$C = \sum_{k=1}^{K} C_k(s), \quad (24)$$

then from Eqs.(21) and (22) we can write separate differential equations for the individual y_k's as

$$D_k \frac{d^2}{dy_k^2} g_k^*(s, y_k) + (\alpha_k + \beta_k)\frac{d}{dy_k}[(y_k - y_k^*) g_k^*(s, y_k)]$$
$$-[s_k + u(s)(R_k - C_k(s))] g_k^*(s, y_k) = 0, \quad k = 1, \cdots, K. \quad (25)$$

where y_k^* is defined by Eq.(6) and s has a partition of

$$s = \sum_{k=1}^{K} s_k. \quad (26)$$

Equation (25), after some changes of variables, becomes a parabolic cylinder function (or Weber's function). Its elementary solutions which have the appropriate behavior as $y_k \to \pm\infty$ require (see [4, 10])

$$(\frac{R_k \sigma_k^*}{\alpha_k + \beta_k})^2 u^2(s) + \frac{C_k(s) - R_k y_k^*}{\alpha_k + \beta_k} u(s) - \frac{s_k}{\alpha_k + \beta_k} = i_k, \quad (27)$$

where $i_k = 0, 1, 2, \cdots$ being a nonnegative integer.

Then the solution of Eq.(25), corresponding to i_k, is given by

$$g_{k,i_k}^*(s, y_k) = e^{-\frac{(y_k - y_k^*)^2}{4\sigma_k^{*2}}} 2^{-\frac{i_k}{2}} e^{-(\frac{y_k - y_k^*}{2\sigma_k^*} - \frac{R_k \sigma_k^*}{\alpha_k + \beta_k} u_{i_k}(s))^2}$$
$$\cdot H_{i_k}(\frac{y_k - y_k^*}{\sqrt{2}\sigma_k^*} - \frac{\sqrt{2} R_k \sigma_k^*}{\alpha_k + \beta_k} u_{i_k}(s)), \quad (28)$$

where $H_{i_k}(\cdot)$ is the Hermite polynomial of order i_k.

From Eqs.(24), (26) and (27) we can see that $u(s)$ must satisfy the following quadratic equation for each integer vector $\mathbf{i} = (i_1, i_2, \cdots, i_K)$:

$$(\sum_{k=1}^{K} \frac{R_k^2 \sigma_k^{*2}}{\alpha_k + \beta_k}) u^2(s) + (C - \sum_{k=1}^{K} R_k y_k^*) u(s)$$
$$-(s + \sum_{k=1}^{K} i_k(\alpha_k + \beta_k)) = 0. \quad (29)$$

Since all the components i_k are nonnegative and the Laplace transform variable s should be considered in the stable right half plane (i.e. $Re\{s\} \ge 0$), Eq.(29) yields two roots: one positive denoted $u_{\mathbf{i}}^+(s)$ (i.e., $Re\{u_{\mathbf{i}}^+(s)\} > 0$), and one negative denoted $u_{\mathbf{i}}^-(s)$ (i.e., $Re\{u_{\mathbf{i}}^-(s)\} < 0$). Needless to say, the explicit expression for $u_{\mathbf{i}}^-(s)$ is readily available by solving the quadratic equation (29). Note that the corresponding roots in the exact transient analysis [8, 9] are only numerically available as functions of s, whereas the diffusion approximation analysis gives closed form expression for the roots $u_{\mathbf{i}}^-(s)$. This significantly reduces our computation efforts.

We assume that the capacity of the statistical multiplexer output buffer is infinite and the system is stable. Then Eq.(23) suggests that an inclusion of any $u_{\mathbf{i}}^+(s)$ would lead to an unstable solution. Thus, the general solution of Eq.(21) is represented as

$$f^*(\mathrm{y}; x, s|\mathrm{y}_0, 0) = . \sum_{i} a_i(s) e^{u_i^-(s)x} \prod_{k=1}^{K} g_{k,i_k}^*(s, y_k). (30)$$

Then the transient probability distribution function that the buffer content exceeds x is given by

$$F^*(s,x) \stackrel{def}{=} \mathcal{L}_t\{Prob(Q(t) > x)\}$$

$$= \int_{x}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{K} f^*(\mathbf{y}; x, s|\mathbf{y}_0, 0)\, dx\, d\mathbf{y}$$
$$= \sum_{\mathbf{i}} \tilde{a}_{\mathbf{i}}(s) e^{u_{\mathbf{i}}^-(s)x}. \quad (31)$$

3 Applications and Discussion

We now consider two cases where we compute the packet loss probabilities by applying the approximate transient analysis method developed in Section 2.

3.1 Aggregate Traffic Behavior

First, we consider the case where there is no buffer or only a small amount of buffer available at the multiplexer to absorb fluctuations at the cell level. If there is no buffer (i.e., a "loss system"), the cells are lost, whenever the aggregate arrival exceeds the output link capacity C. The loss probability $Prob(R(t) > C)$ can be calculated using the error function defined by $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-x^2} dx$:

$$Prob(R(t) > C) = \frac{1}{\sqrt{2\pi\sigma_R^2(t)}} \int_C^{+\infty} e^{-\frac{(x-m_R(t))^2}{2\sigma_R^2(t)}} dx = \frac{1}{2} erfc(\theta(t)), \quad (32)$$

where

$$\theta(t) = \frac{C - m_R(t)}{\sqrt{2}\sigma_R(t)}. \quad (33)$$

Using the inequalities (for $z > 0$):

$$\frac{e^{-z^2}}{z + \sqrt{z^2+2}} < \int_z^{+\infty} e^{-x^2} dx \le \frac{e^{-z^2}}{z + \sqrt{z^2 + \frac{4}{\pi}}}, \quad (34)$$

we obtain the following bounds on $Prob(R(t) > C)$, which is the probability that the aggregate packet arrival rate exceeds the multiplexer output capacity momentarily:

$$LB(t) < Prob(R(t) > C) \le UB(t), \; m_R(t) < C, \quad (35)$$
$$1 - UB(t) < Prob(R(t) > C) \le 1 - LB(t), \; m_R(t) \ge C, \quad (36)$$

where

$$LB(t) = \frac{e^{-\theta^2(t)}}{\sqrt{\pi}} \frac{1}{\theta(t) + \sqrt{\theta^2(t) + 2}} \quad (37)$$

$$UB(t) = \frac{e^{-\theta^2(t)}}{\sqrt{\pi}} \frac{1}{\theta(t) + \sqrt{\theta^2(t) + \frac{4}{\pi}}}. \quad (38)$$

These bounds are asymptotically tight, hence $Prob(R(t) > C) \approx \frac{1}{2\sqrt{\pi}\theta(t)} e^{-\theta^2(t)}$, when $\theta(t)$ is large. The latter quantity coincides with the one obtained by applying the theory of large deviation [1, 3]. Note that $\{R(t) > C\}$ implies $\{Q(t) > 0\}$, thus from Eqs.(35) and (36) we obtain the following lower bound on $Prob(Q(t) > 0)$, if no or small buffer capacity is available:

$$Prob(Q(t) > 0) \ge Prob(R(t) > C) \begin{cases} > LB(t), & m_R(t) < C, \\ \ge 1 - UB(t), & m_R(t) \ge C. \end{cases} \quad (39)$$

3.2 Asymptotic Buffer Behavior

Next, we consider the case where a medium or large buffer capacity is available. We can evaluate the transient loss probability at the burst level by using Eq.(31). Here we assume that the buffer has an infinite capacity and the cell loss probability for a system with a finite buffer capacity B is approximated by $Prob(Q(t) > B)$. Unfortunately, there is no simple way to determine the unknown coefficients $\tilde{a}_{\mathbf{i}}(s)$ of Eq.(31). However, the asymptotic behavior (i.e., for sufficiently large B) is characterized by the exponential term that has the largest negative (i.e., dominant) root. In our case, this dominant root is $u_0^-(s)$ which corresponds to $\mathbf{i} = (0, 0, \cdots, 0)$. Thus, we find

$$F^*(s, B) \approx \frac{Prob(R(t) > C)}{s} e^{u_0^-(s)B}, \quad \text{for large } B. \quad (40)$$

The unknown coefficient of the exponential term is estimated from loss probability at $B = 0$, which we find in Eq.(39).

Taking the inverse Laplace transform on (40), we have

$$Prob(Q(t) > B) \approx Prob(R(t) > C) \cdot e^{-\frac{\zeta}{\eta}B} \cdot \int_0^t e^{-\frac{\zeta^2}{4\eta}t'} \frac{B}{2t'\sqrt{\eta\pi t'}} e^{-\frac{B^2}{4\eta t'}} dt' \quad (41)$$

where

$$\zeta = C - \sum_{k=1}^{K} R_k y_k^*, \quad \text{and} \quad \eta = \sum_{k=1}^{K} \frac{R_k^2 \sigma_k^{*2}}{\alpha_k + \beta_k}. \quad (42)$$

The integrand is a function of t with a bell shape centered around the time

$$t = \frac{-3\eta + \sqrt{\zeta^2 B^2 + 9\eta^2}}{\zeta^2} \approx \frac{B}{\zeta} = \frac{B}{C - E[R]}, \quad \text{for large } B. \quad (43)$$

Eq.(43) suggests that the transient period (or the relaxation time) for asymptotic buffer behavior depends on the load (traffic intensity) of the system but its expression becomes rather simple when the buffer capacity increases. We also notice that when the traffic intensity ($\rho = \frac{E[R]}{C}$) is fixed, multiplexing more traffic sources will shorten the queue transient period. This agrees with our earlier observation in [13].

3.3 Numerical Results

Some numerical examples are given as follows:

We choose the following parameters, which are same as the ones discussed in Kosten [12] and our previous paper [10]: i.e., $K = 2$, $C = 38$,

$$\text{Type 1 Sources}: N_1 = 25,\ \alpha_1 = 0.4,\ \beta_1 = 1.5,\ R_1 = 2\ .$$

$$\text{Type 2 Sources}: N_2 = 50,\ \alpha_2 = 0.6,\ \beta_2 = 0.75,\ R_2 = 1\ .$$

In Figure 4, the transient upper and lower bounds of $Prob(R(t) > C)$ (see Eqs.(35) and (36)) are shown for two cases: The first is a case where the system is overloaded—we assume that 20 Type 1 sources (out of 25) and 30 Type 2 sources (out of 50) are initially "on", i.e., $Y_1(0) = 20$, and $Y_2(0) = 30$. The second is a case in which the system is initially underloaded, i.e., $Y_1(0) = 10$, and $Y_2(0) = 10$. In both cases both upper and lower bounds of $Prob(R(t) > C)$ approach their steady-state values well before $t = 4$. The time-constant (or relaxation time), which characterizes the convergent rate of $\theta(t)$ (thus the transient loss probability $Prob(R(t) > C)$), is closely related to the inverse of $\min_k\{b_1, b_2, \cdots, b_K\}$ (see Eqs.(6) and (19)). In this example, it is $b_2^{-1} = (\alpha_2 + \beta_2)^{-1} = 0.74$.

In Figure 5, we show that buffer overflow probability (in logarithm scale) vs. the buffer capacity B, by taking the inverse Laplace transform of Eq.(40). The initial condition we set corresponds to the second case in Figure 4, i.e., the initially underloaded case ($Y_1(0) = 10$, $Y_2(0) = 10$). We further assume here that the buffer is initially empty. The different curves correspond to different time instants. Unlike the result of Figure 4, however, the steady-state cannot be achieved until after time $t > 40$ elapses. In other words, the time constant of buffer overflow probability is almost an order of magnitude larger than that of the on-off source behavior. This significant difference between the time constants of the two probability functions (associated with the diffusion process $R(t)$ and the fluid approximation $Q(t)$, respectively) is certainly attributable to Eq.(2), i.e., the process $Q(t)$ is an integration of $R(t)$.

4 Conclusion

To summarize, we have developed a multi-dimensional diffusion model to characterize the transient behavior of an ATM statistical multiplexer with multiple types of traffic. The time-varying aggregate traffic is approximated by the Ornstein-Uhlenbeck process, which adequately captures the essence of the correlated nature of the bursty arrival process. We have shown some simulated sample paths and their correlograms to justify our diffusion approximation method. We are presently investigating the analytic results and the diffusion process in terms of their power spectrum (or the periodogram) [6], and the results will be reported elsewhere.

The transient cell loss probabilities and time constants are estimated at both cell and burst levels in our examples. Through these examples, we have found that while the time constant for the transient behavior of the on-off sources can be estimated from those of the covariance function of $R(t)$ (see Eq.(19)), the behavior of the buffer overflow probability (i.e., packet or cell loss probability) is governed by the transient behavior of the dominant root $u_0^-(s)$, and its time constant depends, as shown in (43), on the overall traffic load and number of multiplexed sources.

Note that the method we developed in this paper is applicable to more general traffic source models than assumed in exact analyses [8, 9]. For example, the exponential distribution assumption is not necessary when N_k is sufficiently large.

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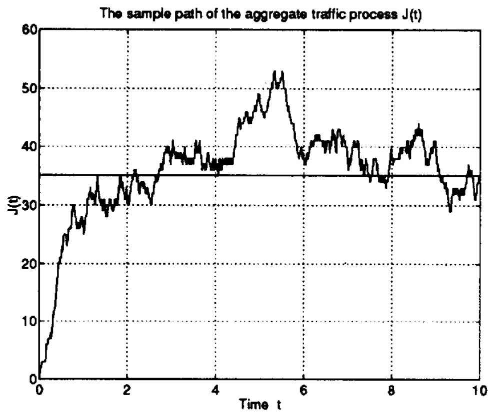

Figure 1: The sample path of the aggregate traffic process $J(t)$ with $N = 100$, $\alpha = 0.385$ and $\beta = 0.71$.

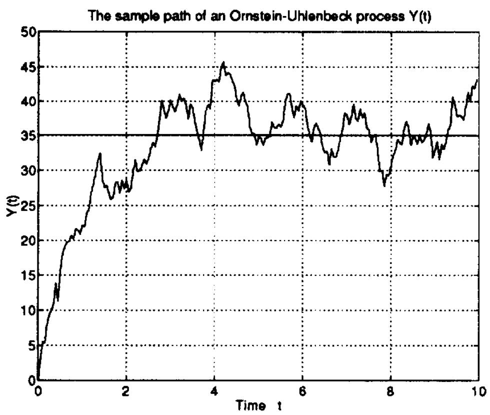

Figure 2: The sample path of an Ornstein-Uhlenbeck process $Y(t)$ with parameters defined in Eq.(6), where $N = 100$, $\alpha = 0.385$ and $\beta = 0.71$.

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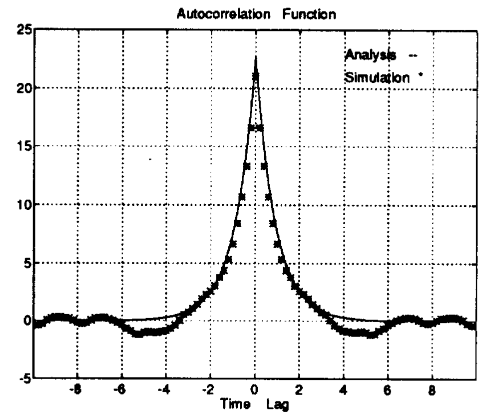

Figure 3: The autocorrelation functions of diffusion process $Y(t)$: analytical results vs. simulation results. $N = 100$, $\alpha = 0.385$ and $\beta = 0.71$.

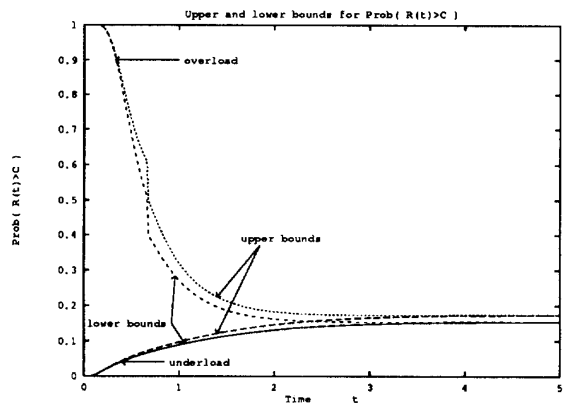

Figure 4: Upper and lower bounds of $Prob(R(t) > C)$. The initial conditions are: $Y_1(0) = 20$, $Y_2(0) = 30$ and $Y_1(0) = 10$, $Y_2(0) = 10$, respectively.

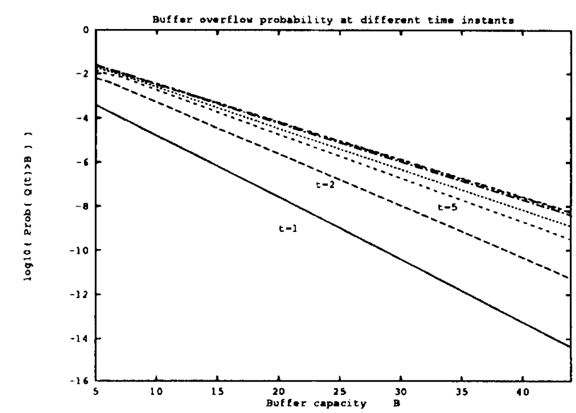

Figure 5: $log_{10}(Prob(Q(t) > B))$ vs. buffer capacity B. The time instants are $1, 3, 5, 10, 40, 100$ and 200, respectively.